The Auslander-Reiten conjecture for certain non-Gorenstein Cohen-Macaulay rings

WVU Algebra Seminar via Zoom

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§4. Introduction 2.

Def. We say that R satisfies (ARC) if, for any f.g. R-mod. M,
$\operatorname{Ext}_R^{>0}(M, M \oplus R) = 0 \Rightarrow M$ is proj.

Setting
- $(R, \mathfrak{m})$: Noeth. local ring.
- M : f.g. R-mod.
- Q : ideal of R generated by an R-reg. seq.

Fact $R : (\mathrm{ARC}) \iff R/Q : (\mathrm{ARC})$

Question For $\ell > 0$, $R : (\mathrm{ARC}) \iff R/Q^\ell : (\mathrm{ARC})$?

Motivations of Question :

(0) If $\ell > 1$, then R/Q^ℓ is neither a Gorenstein ring nor a domain.

(1) Let
- S : CM local ring.
- $\{x_{ij}\}_{\substack{1 \le i \le m \\ 1 \le j \le n}}$: reg. seq. of S. $(m \le n)$.

Set
- $I = I_m(x_{ij})$: determinantal ideal

  (i.e. I is the ideal of S generated by m-minors of the $m \times n$ matrix (x_{ij}))
- $R = S/I$.

Then

Fact (i) [Eagon - Northcott]

R is a CM ring.

(ii) $\exists Q$: par. ideal of R and $\exists \ell > 0$

s.t. $R/Q \cong S'/Q'^{\ell}$, where

- $S' = S/(\text{reg. seq. of } S)$ and
- Q' : par. ideal of S'

Ex Let

- $S = k[[x_1, x_2, \cdots, x_6]]$.
- $I = I_2 \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}$

$= (x_1x_5 - x_2x_4,\ x_1x_6 - x_3x_4,\ x_2x_6 - x_3x_5)$.

Set

- $R = S/I$
- $Q = (x_1, x_4, x_2 - x_4, x_3 - x_5)$.

Then $R/QR \cong S'/(x_2, x_3)^2 S'$, where $S' = S/Q$.

(2) Let $(R, \mathfrak{m})$ be a CM local ring.

Then

$$v(R) - \dim R + 1 \leq e(R)$$

holds. We say R has min. multi.

(or max. emb. dim.)

if "=" holds.

<u>Fact</u> (Sally)

(i) If $\exists Q \subset \mathfrak{m}$: par. red. (e.g. $|R/\mathfrak{m}| = \infty$),

then R has min. multi. $\Leftrightarrow$ $\mathfrak{m}^2 = Q\mathfrak{m}$.

(ii) If $\mathfrak{m}^2 = Q\mathfrak{m}$ for $\exists Q$: par. red. and

$\exists$ $(S, \mathfrak{n}) \twoheadrightarrow (R, \mathfrak{m})$: ring. hom,

RLR of dim $v(R)$.

(e.g. R is complete)

then $R/Q \cong S/\mathfrak{n}^2$.

(3) Let
- R : CM local ring of dim d.
- $Q = (a_1, \cdots, a_d)$: par. ideal of R
- $R[Qt]$: Rees algebra.

Then

<u>Fact</u> [Barshy]

$$R[Qt] \simeq R[x_1, \cdots, x_d] \Big/ I_2\begin{pmatrix} x_1 & \cdots & x_d \\ a_1 & \cdots & a_d \end{pmatrix}$$

(4) Let
- R : CM local ring

Then

<u>Fact</u> [Herzog]

$$R[t^a, t^b, t^c] \simeq \begin{cases} R[X, Y, Z] \big/ (\text{reg. seq. of } R[X, Y, Z]) & \text{or} \\ R[X, Y, Z] \Big/ I_2\begin{pmatrix} X^{\alpha_1} & Y^{\alpha_2} & Z^{\alpha_3} \\ Y^{\beta_2} & Z^{\beta_3} & X^{\beta_1} \end{pmatrix} \end{cases}$$

§5. Main theorem.

Thm [K]

Let
- $(R, \mathfrak{m})$: Gorenstein local ring.
- $x_1, \cdots, x_n$: reg. seq. of R.
- $Q = (x_1, \cdots, x_n)$.
- $\ell > 0$.

Consider the following conditions.

(1) R : (ARC).

(2) R/Q^ℓ : (ARC).

Then (2) $\Rightarrow$ (1) holds.

(1) $\Rightarrow$ (2) holds if $\ell \leq n$.

Lem Let
- $(R, \mathfrak{m})$: Noeth. local ring
- I : $\mathfrak{m}$-primary ideal of R
- M : f.g. R-mod.
- N : f.g. R/I-mod.

(1) Suppose that R/I is Gorenstein.

Then $\mathrm{Ext}_R^{>0}(M, R/I) = 0 \Rightarrow \mathrm{Tor}_{>0}^R(M, R/I) = 0$

(2) $\mathrm{Tor}_{>0}^R(M, R/I) = 0$

$\Rightarrow \mathrm{Ext}_R^i(M, N) \simeq \mathrm{Ext}_{R/I}^i(M/IM, N)$ for $\forall i \in \mathbb{Z}$.

Proof of Lem)

Let $F_\bullet \to M \to 0$ be a min. R-free res. of M.

(1): It follows from

$$\begin{array}{ccc} 0 \to \mathrm{Hom}_R(M, R/I) & \longrightarrow & \mathrm{Hom}_R(F_\bullet, R/I) \\ \cong & \circlearrowright & \cong \\ 0 \to \mathrm{Hom}_{R/I}(M/IM, R/I) & \to & \mathrm{Hom}_{R/I}(F_\bullet/IF_\bullet, R/I) \end{array}.$$

(2): It follows from

$$\begin{array}{ccc} 0 \to \mathrm{Hom}_R(M, N) & \longrightarrow & \mathrm{Hom}_R(F_\bullet, N) \\ \cong & \circlearrowright & \cong \\ 0 \to \mathrm{Hom}_{R/I}(M/IM, N) & \to & \mathrm{Hom}_{R/I}(F_\bullet/IF_\bullet, N) \end{array}.$$

Proof of Thm:

For each implications, we may assume that Q is a par. ideal of R.

(If $\dim R/Q > 0$, $\exists\, a \in R$: NZD of R/Q.
Then, by passing to

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & R/aR \arrow[d] \\ R/Q^\ell \arrow[r] & R/aR+Q^\ell \end{tikzcd},$$

we may assume $\dim R/Q = 0$.)

We may also assume that $n \geq 2$ and $\ell \geq 2$.

Our main tool is the following ex. seq.

$$(\star)\quad \begin{cases} 0 \to Q/Q^2 \to R/Q^2 \to R/Q \to 0 \\ 0 \to Q^2/Q^3 \to R/Q^3 \to R/Q^2 \to 0 \\ \qquad\qquad \vdots \\ 0 \to Q^{\ell-2}/Q^{\ell-1} \to R/Q^{\ell-1} \to R/Q^{\ell-2} \to 0 \\ 0 \to Q^{\ell-1}/Q^{\ell} \to R/Q^{\ell} \to R/Q^{\ell-1} \to 0 . \end{cases}$$

Note that $Q^i/Q^{i+1} \simeq (R/Q)^{\oplus \binom{i+n-1}{n-1}}$ for $i \geq 0$.

(2) $\Rightarrow$ (1) :

$$\boxed{\begin{array}{l}\underline{\text{WTS:}}\quad M : \text{f.g. } R\text{-mod. s.t. } \operatorname{Ext}_R^{>0}(M, M \oplus R) = 0 \\ \qquad \Rightarrow M : R\text{-free}\end{array}}$$

$$\boxed{\underline{\text{Claim}}\quad \operatorname{Ext}_R^{>0}(M, M/QM \oplus R/Q) = 0} \quad \cdots ①$$

Proof of Claim) Apply $\operatorname{Hom}_R(M, -)$ to

$$\begin{cases} 0 \to R \xrightarrow{x_1} R \to R/x_1R \to 0 \\ \qquad \vdots \\ 0 \to R/(x_1, \cdots, x_{n-1}) \xrightarrow{x_n} R/(x_1, \cdots, x_{n-1}) \to R/Q \to 0. \end{cases}$$

Then $\operatorname{Ext}_R^{>0}(M, R/Q) = 0$

Similarly, we have $\operatorname{Ext}_R^{>0}(M, M/QM) = 0$ since M is a MCM R-mod. //

① $\overset{\text{lem (1)}}{\rightsquigarrow}$ $\operatorname{Tor}^R_{>0}(M, R/Q) = 0$

$\overset{(\star)}{\rightsquigarrow}$ $\operatorname{Tor}^R_{>0}(M, R/Q^i) = 0$ for $\forall i > 0$ $\cdots ②$

(i.e., M is a lifting of M/Q^iM)

Hence, we get

$$
\star\star
\begin{cases}
0 \to (M/QM)^{\oplus n} \to M/Q^2M \to M/QM \to 0 \\
0 \to (M/QM)^{\oplus \binom{2+n-1}{n-1}} \to M/Q^3M \to M/Q^2M \to 0 \\
\qquad\qquad\vdots \\
0 \to (M/QM)^{\oplus \binom{\ell-2+n-1}{n-1}} \to M/Q^{\ell-1}M \to M/Q^{\ell-2}M \to 0 \\
0 \to (M/QM)^{\oplus \binom{\ell-1+n-1}{n-1}} \to \underset{\displaystyle M}{\underset{\shortparallel}{M/Q^{\ell}M}} \to M/Q^{\ell-1}M \to 0 .
\end{cases}
$$

①
$\mathrm{Hom}_R(M,-)$

$$\rightsquigarrow \quad \mathrm{Ext}_R^{>0}(M, M/Q^{\ell}M) = 0$$

Similarly, we obtain $\mathrm{Ext}_R^{>0}(M, R/Q^{\ell}) = 0$.

Thus, $\mathrm{Ext}_R^{>0}(M, M/Q^{\ell}M \oplus R/Q^{\ell}) = 0$

lem(2)
②
$$\rightsquigarrow \quad \mathrm{Ext}_{R/Q^{\ell}}^{>0}(M/Q^{\ell}M, M/Q^{\ell}M \oplus R/Q^{\ell}) = 0$$

R/Q^{ℓ} : (ARC)
$$\rightsquigarrow \quad M/Q^{\ell}M : R/Q^{\ell}\text{-free}$$

②
$$\rightsquigarrow \quad M : R\text{-free} \qquad \blacksquare$$

(1) $\Rightarrow$ (2): Set $R_i := R/Q^i$ for $i > 0$.

$$\boxed{\begin{array}{l}\underline{\text{WTS}}: N : \text{f.g. } R_\ell\text{-mod. s.t. } \operatorname{Ext}^{>0}_{R_\ell}(N, N \oplus R_\ell) = 0 \\ \qquad \Rightarrow N : R_\ell\text{-free}\end{array}}$$

$$☆ \left\{\begin{array}{l}
0 \to R_1^{\oplus n} \to R_2 \to R_1 \to 0 \\
\qquad \vdots \\
0 \to R_1^{\oplus \binom{\ell-3+n-1}{n-1}} \to R_{\ell-2} \to R_{\ell-3} \to 0 \\
0 \to R_1^{\oplus \binom{\ell-2+n-1}{n-1}} \to R_{\ell-1} \to R_{\ell-2} \to 0 \\
0 \to R_1^{\oplus \binom{\ell-1+n-1}{n-1}} \to R_\ell \to R_{\ell-1} \to 0
\end{array}\right. \quad \Big\downarrow \operatorname{Hom}_{R_\ell}(N, -)$$

☆☆☆

- $\cdots \to \operatorname{Ext}^j_{R_\ell}(N, R_1)^{\oplus n} \to \operatorname{Ext}^j_{R_\ell}(N, R_2) \to \operatorname{Ext}^j_{R_\ell}(N, R_1)$

  $\to \operatorname{Ext}^{j+1}_{R_\ell}(N, R_1)^{\oplus n} \to \cdots$

$$\vdots$$

- $\cdots \to \operatorname{Ext}^j_{R_\ell}(N, R_1)^{\oplus \binom{\ell-3+n-1}{n-1}} \to \operatorname{Ext}^j_{R_\ell}(N, R_{\ell-2}) \to \operatorname{Ext}^j_{R_\ell}(N, R_{\ell-3})$

  $\to \operatorname{Ext}^{j+1}_{R_\ell}(N, R_1)^{\oplus \binom{\ell-3+n-1}{n-1}} \to \cdots$

- $$\cdots \to \mathrm{Ext}^j_{R_\ell}(N, R_1)^{\oplus \binom{\ell-2+n-1}{n-1}} \to \mathrm{Ext}^j_{R_\ell}(N, R_{\ell-1}) \to \mathrm{Ext}^j_{R_\ell}(N, R_{\ell-2})$$
$$\to \mathrm{Ext}^{j+1}_{R_\ell}(N, R_1)^{\oplus \binom{\ell-2+n-1}{n-1}} \to \cdots$$ and

- $$\mathrm{Ext}^j_{R_\ell}(N, R_{\ell-1}) \simeq \mathrm{Ext}^{j+1}_{R_\ell}(N, R_1)^{\oplus \binom{\ell-1+n-1}{n-1}}$$ for $\forall j > 0$.

Set $E_j := \ell_{R_\ell}(\mathrm{Ext}^j_{R_\ell}(N, R_1))$ for $j > 0$.

Then

$$\binom{\ell-1+n-1}{n-1} \cdot E_{j+1} = \ell_{R_\ell}(\mathrm{Ext}^j_{R_\ell}(N, R_{\ell-1}))$$
$$\leq \binom{\ell-2+n-1}{n-1} \cdot E_j + \underbrace{\ell_{R_\ell}(\mathrm{Ext}^j_{R_\ell}(N, R_{\ell-2}))}_{\geq}$$
$$\binom{\ell-3+n-1}{n-1} \cdot E_j + \ell_{R_\ell}(\mathrm{Ext}^j_{R_\ell}(N, R_{\ell-3}))$$
$$\vdots$$

$$\leq \left(\sum_{i=0}^{\ell-2} \binom{i+n-1}{n-1} \right) \cdot E_j$$

$$= \binom{\ell-2+n}{n} \cdot E_j .$$

Hence $$E_{j+1} \leq \frac{\binom{\ell-2+n}{n}}{\binom{\ell-1+n-1}{n-1}} E_j$$

$$= \underbrace{\frac{\ell-1}{n}}_{\substack{\wedge \\ 1}} E_j \quad \text{for } \forall j > 0.$$

by the assumption.

$$\rightsquigarrow E_{j+1} = 0 \quad \text{for } j \gg 0.$$

On the other hand, by (***), we have

$$E_{j+1} = 0 \Rightarrow E_j = 0 \quad \text{for } \forall j > 0.$$

Therefore, $E_j = 0$ for $\forall j > 0$

i.e. $$\operatorname{Ext}^{>0}_{R_\ell}(N, R_1) = 0$$

lem (1)

$\rightsquigarrow$ $\mathrm{Tor}^{R_\ell}_{>0}(N, R_1) = 0$ $\cdots$ ③

(i.e. N is a lifting of N/QN).

⊛ $\} \mathrm{Hom}_{R_\ell}(N, -)$

$\rightsquigarrow$ $\mathrm{Tor}^{R_\ell}_{>0}(N, R_i) = 0$ for $\forall\, 1 \leq i \leq \ell-1$.

i.e. we have

$$
⊛⊛ \begin{cases}
0 \to (M/QM)^{\oplus n} \to M/Q^2M \to M/QM \to 0 \\
0 \to (M/QM)^{\oplus \binom{2+n-1}{n-1}} \to M/Q^3M \to M/Q^2M \to 0 \\
\vdots \\
0 \to (M/QM)^{\oplus \binom{\ell-2+n-1}{n-1}} \to M/Q^{\ell-1}M \to M/Q^{\ell-2}M \to 0 \\
0 \to (M/QM)^{\oplus \binom{\ell-1+n-1}{n-1}} \to \underset{\displaystyle M}{\overset{\displaystyle M/Q^\ell M}{\parallel}} \to M/Q^{\ell-1}M \to 0
\end{cases}
$$

by applying $- \otimes_{R_\ell} N$ to ⊛.

similarly to $\mathrm{Ext}^{>0}_{R_\ell}(N, R_1) = 0$

$\rightsquigarrow$ $\mathrm{Ext}^{>0}_{R_\ell}(N, N/QN) = 0$

③ and lem(2)

$\rightsquigarrow$ $\mathrm{Ext}^{>0}_{R_1}(N/QN, N/QN) = 0$

R/Q : (ARC)

$\rightsquigarrow$ N/QN is R/Q-free

③

$\rightsquigarrow$ N is R-free $\square$

Applycations of Theorem

Let
- $(R,\mathfrak{m})$: Gor. normal domain.
- Q : par. ideal of R.

(1) The determinantal ring

$$R[X_{ij}]_{1\le i\le m,\, 1\le j\le n} \big/ I_m(X_{ij})$$

satisfies (ARC) if $2m \le n+1$.

(2) The Rees alg. $R[\![Qt]\!]$

satisfies (ARC).

(3) $R[\![t^a, t^b, t^c]\!]$ satisfies (ARC).

§6. generalization of Sally's result.

Let $\circ$ $(R, \mathfrak{m})$: CM local ring

$\circ$ I : $\mathfrak{m}$-primary ideal

Def [Goto-Ozeki-Takahashi-Watanabe-Yoshida].

I is called an Ulrich ideal if

$$\begin{cases} \circ\ I^2 = qI \text{ for } {}^{\exists} q \subset I : \text{par. ideal} \\ \circ\ I/I^2 \text{ is } R/I\text{-free} \end{cases}$$

Thm (K)

Suppose ${}^{\exists}I$: Ulrich ideal s.t. R/I is a complete intersection.

then ${}^{\exists}$ $R/q \simeq S/Q^2$: isom. of rings, where

$$\begin{cases} q : \text{par. red. of } I. \\ S : \text{local complete intersection} \\ Q : \text{par. ideal of } S. \end{cases}$$

In particular,

$$\mathfrak{m}^2 = \mathfrak{q}\mathfrak{m} \Longrightarrow R/\mathfrak{q} \simeq S/Q^2$$

$$\Longrightarrow R : \text{(ARC)}.$$

Proof of Thm)

By passing to $R \to R/\mathfrak{q}$, we may assume that $\dim R = 0$ and $\mathfrak{q} = 0$.

Then $\exists\, \varphi : \underset{\text{RLR of dim } v := v(R)}{(S, \mathfrak{n})} \twoheadrightarrow (R, \mathfrak{m})$: ring hom.

Set $\circ$ $\overline{\varphi} : S \overset{\varphi}{\twoheadrightarrow} R \overset{\varepsilon}{\twoheadrightarrow} R/I$

$\circ$ $J := \operatorname{Ker} \overline{\varphi}$.

Since $S/J \simeq R/I$: c.i., $J = (x_1, \cdots, x_v)$.

On the other hand, since $I \simeq \big(\underbrace{R/I}_{\text{c.i.}}\big)^{\oplus \mu_R(I)}$,

$$I = (0) :_R I \simeq \operatorname{Hom}_R(R/I, R).$$

It follows that

$$\mu_R(I) = r_R(I) = r_R(\operatorname{Hom}_R(R/I, R)) = \underset{=:r}{r(R)}.$$

Hence $\quad I = JR = (\overline{x_1}, \cdots, \overline{x_v})$

$$= (\overline{x_1}, \cdots, \overline{x_r})$$

after renumbering of $x_1, \cdots, x_v$.

Thus $\quad J = (x_1, \cdots, x_r) + \operatorname{Ker}\varphi.$

$\parallel$

$(x_1, \cdots, x_r, x_{r+1}, \cdots, x_v)$

For $r+1 \le {}^\forall i \le v$, write

$$x_i = \sum_{j=1}^{r} \underbrace{c_{ij}}_{\in R} x_j + \underbrace{y_i}_{\in \operatorname{Ker}\varphi}.$$

Set $\quad X := (x_1, \cdots, x_r)$

$\quad\quad\; Y := (y_{r+1}, \cdots, y_v).$

Then

$$\begin{array}{c} J = X + \operatorname{Ker}\varphi = X + Y \\ \cup \\ \operatorname{Ker}\varphi \\ \cup \\ J^2 + Y = X^2 + Y \end{array}$$

$\ell_R(I) = r \cdot \ell_R(R/I)$

$\ell_S(X+Y/X^2+Y) = r \cdot \ell_S(S/J).$

since $I^2 = 0$

Therefore, $\operatorname{Ker}\varphi = \mathfrak{J}^2 + \mathcal{Y} = \mathfrak{X}^2 + \mathcal{Y}$.

$$= (x_1, \cdots, x_r)^2 + (y_{r+1}, \cdots, y_v),$$

i.e. $R \simeq S/(x_1, \cdots, x_r)^2 + (y_{r+1}, \cdots, y_v)$

$\boxtimes$